The Theory of Traces for Systems with Probability, Nondeterminism, and Termination

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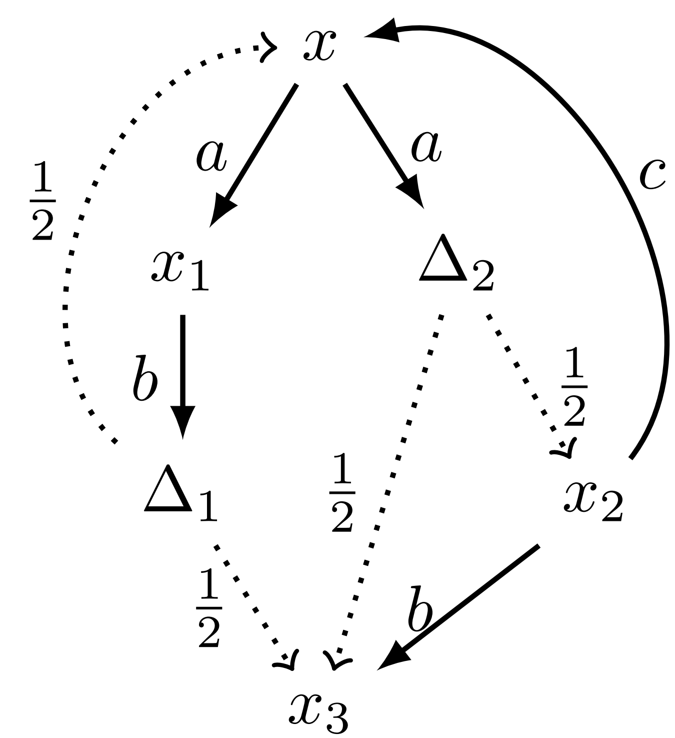
Probabilistic Nondeterministic Labeled Transition Systems

$$t\colon X \to (\mathcal{P}\mathcal{D}X)^A$$

Trace Semantics for these systems is usually defined by means of schedulers and resolutions

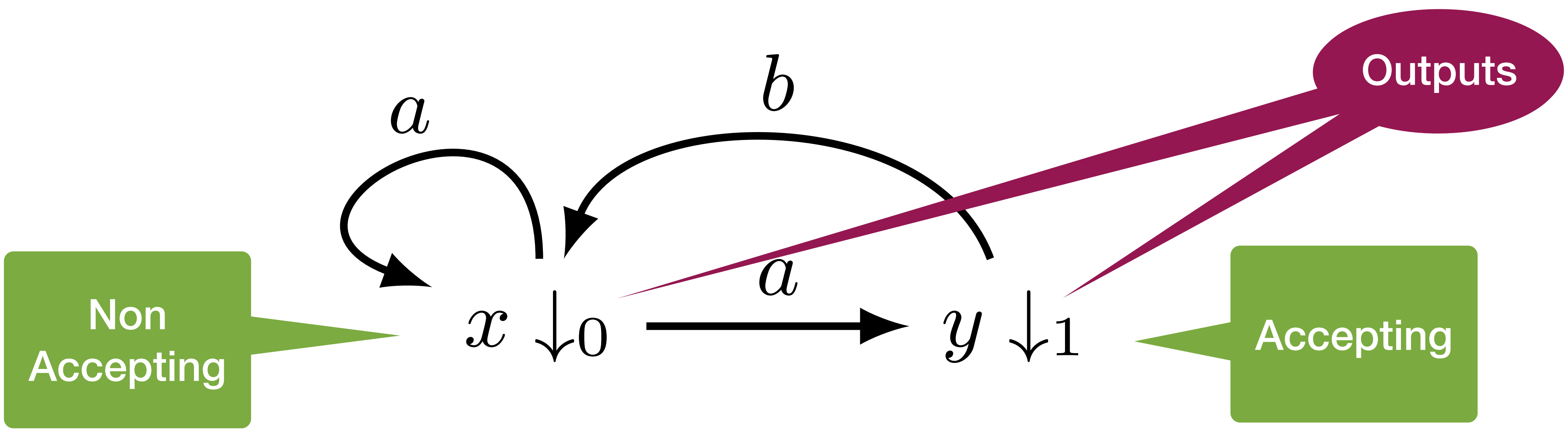


We take a totally different view: our semantics is based on automata theory, algebra and coalgebra

WARNING: In this talk, we will present our theory in its simplest possible form, throwing away all category theory

Nondeterministic Automata

$$\langle o, t\rangle \colon X \;\to\; 2 \times (\mathcal{P}X)^A$$



$$X = \{x, y\} \qquad A = \{a, b\}$$

Language Semantics

NFA = LTS + output

$X \to 2 \times (\mathcal{P}X)^A$

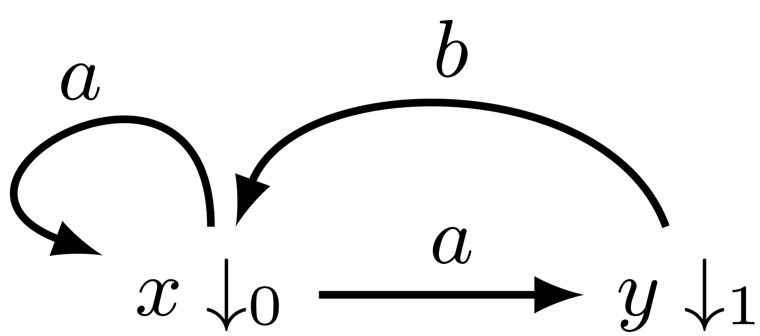
$$\begin{tikzcd} x \downarrow_0 \arrow[loop above, "{a,b}"] \arrow[r, "b"] & y \downarrow_1 \end{tikzcd}$$

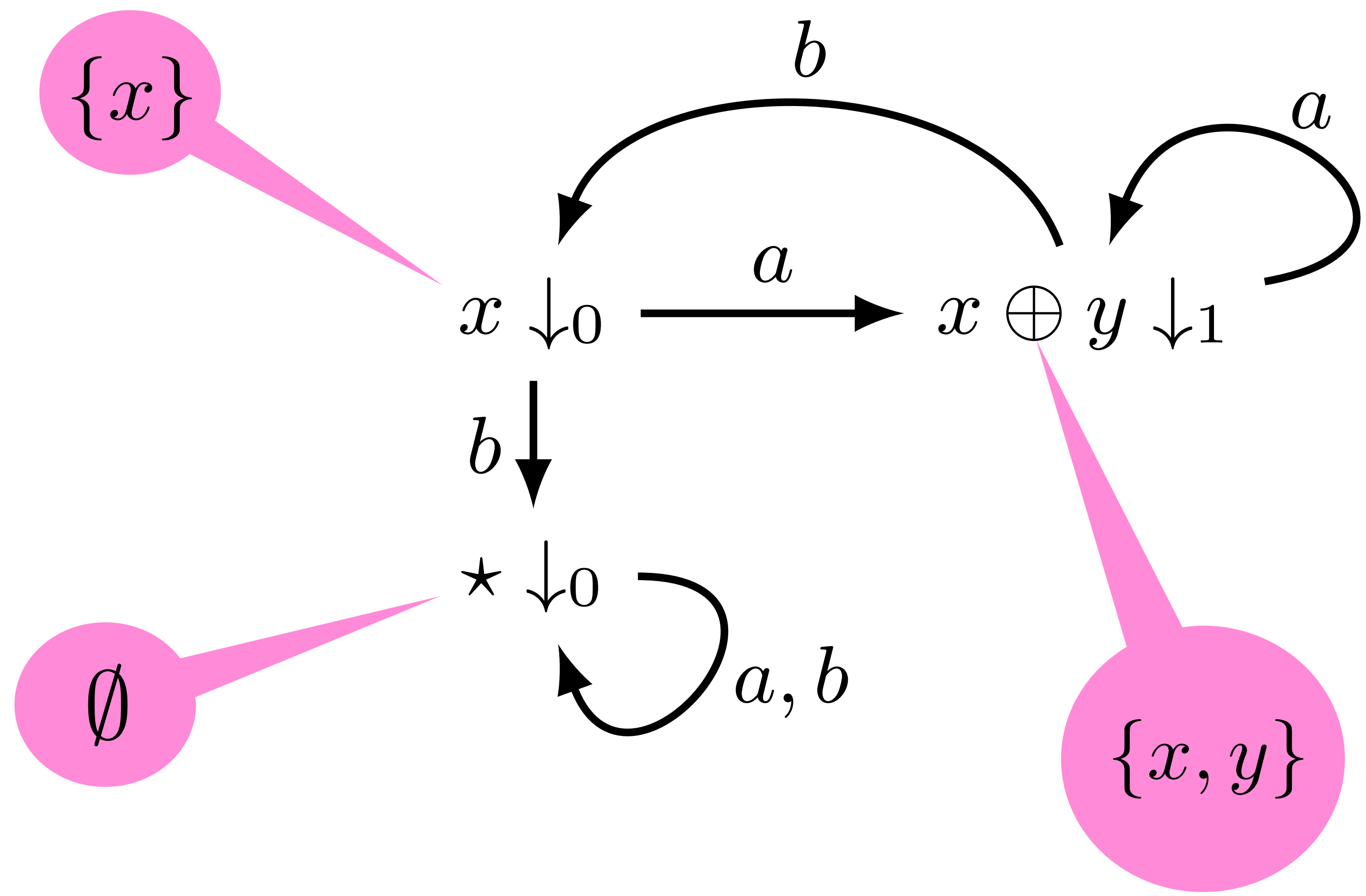
$$[\![\cdot]\!] : X \to 2^{A^*}$$

$$[\![x]\!] = (a \cup b)^* b = \{ w \in \{a, b\}^* \mid w \text{ ends with a } b \}$$

Determinisation for Nondeterministic Automata

$$\langle o, t\rangle \colon X \to 2 \times (\mathcal{P}X)^A \quad \Longrightarrow \quad \langle o^\sharp, t^\sharp\rangle \colon \mathcal{P}X \to 2 \times (\mathcal{P}X)^A$$



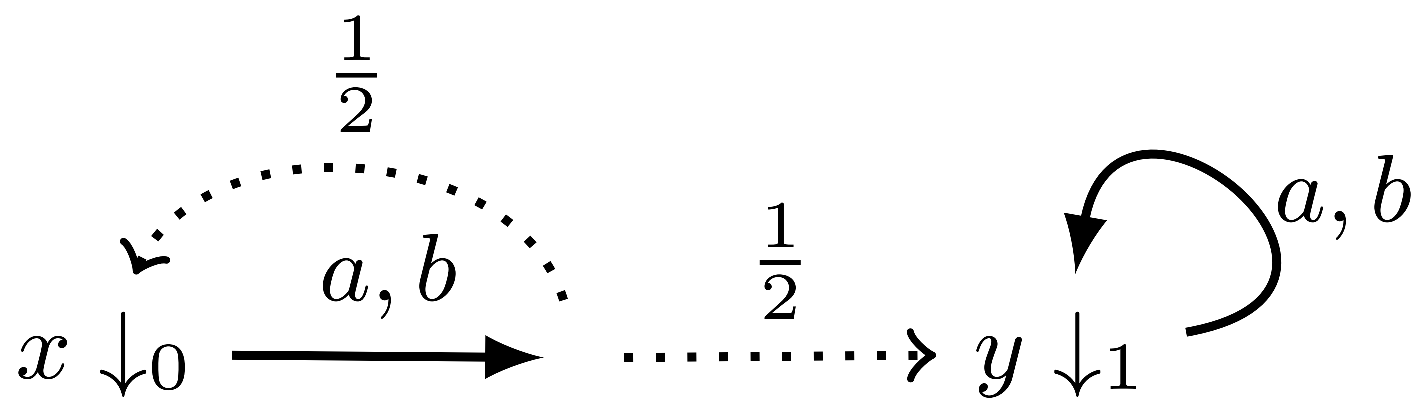


$$[\![\cdot]\!] \colon \mathcal{P}X \to 2^{A^*}$$

$$[\![S]\!](\varepsilon) = o^\sharp(S)$$
$$[\![S]\!](aw) = [\![t^\sharp(S)(a)]\!](w)$$

Probabilistic Automata

$$\langle o, t \rangle \colon X \to [0,1] \times (\mathcal{D}X)^A$$

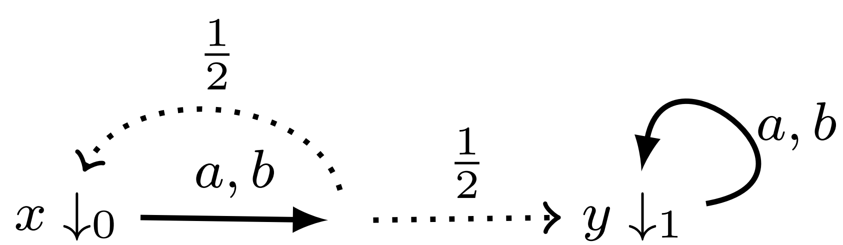


$$X = \{x, y\} \qquad A = \{a, b\}$$

Probabilistic Language Semantics

Rabin PA = PTS + output

$X \to [0,1] \times (\mathcal{D}X)^A$

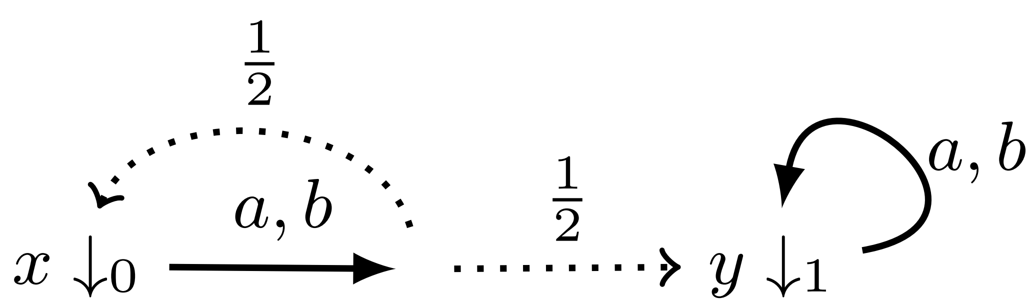


$[\![\cdot]\!] : X \to [0,1]^{A^*}$

$[\![x]\!] = (a \mapsto \frac{1}{2}, aa \mapsto \frac{3}{4}, \ldots)$

Determinisation for Probabilistic Automata

$$\langle o, t \rangle \colon X \to [0,1] \times (\mathcal{D}X)^A \quad \Longrightarrow \quad \langle o^\sharp, t^\sharp \rangle \colon \mathcal{D}X \to [0,1] \times (\mathcal{D}X)^A$$



$$x \downarrow_0$$

δ_x

$$\Big\downarrow a, b$$

$$x +_{\frac{1}{2}} y \downarrow_{\frac{1}{2}}$$

$$\Big\downarrow a, b$$

$$x +_{\frac{1}{4}} y \downarrow_{\frac{3}{4}}$$

$x \mapsto \frac{1}{4}$
$y \mapsto \frac{3}{4}$

$$\Big\downarrow a, b$$

$$\vdots$$

$$[\![\cdot]\!] \colon \mathcal{D}X \to [0,1]^{A^*}$$

$$[\![\Delta]\!](\varepsilon) = o^\sharp(\Delta)$$
$$[\![\Delta]\!](aw) = [\![t^\sharp(\Delta)(a)]\!](w)$$

Toward a GSOS semantics

In the determinisation of **nondeterministic** automata we use terms built of the following syntax

$$s, t \ ::= \ \star, \ s \oplus t, \ x \in X$$

to represent states in $\mathcal{P}X$

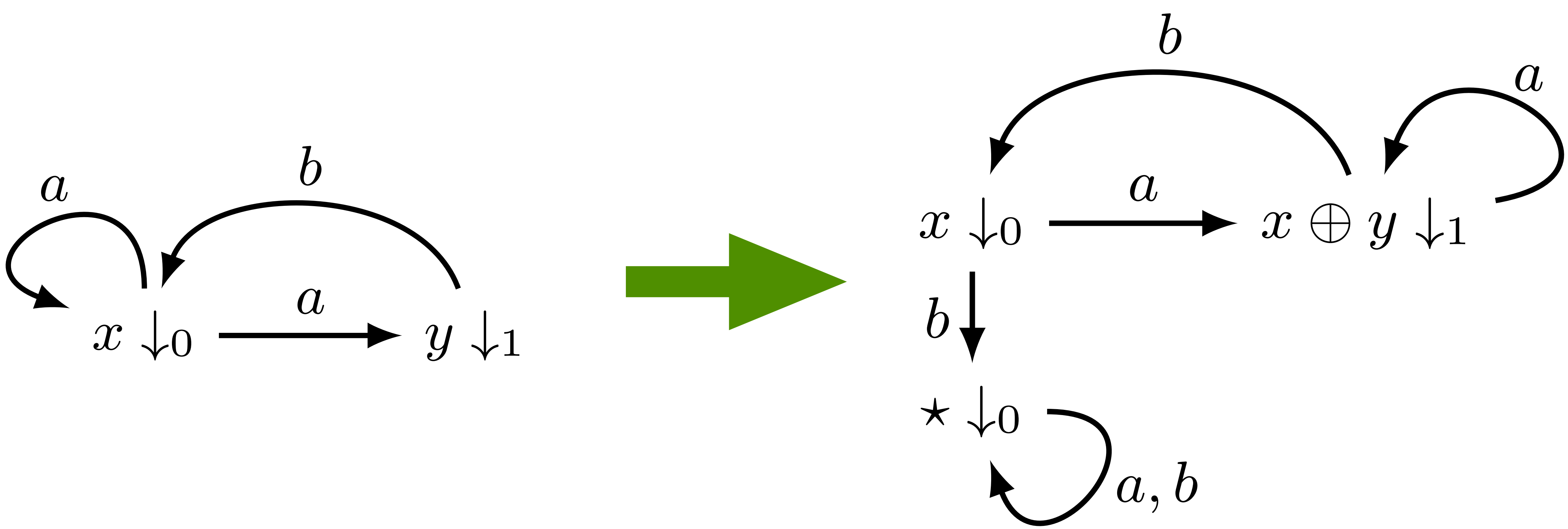
In the determinisation of **probabilistic** automata we use terms built of the following syntax

$$s, t \ ::= \ s +_p t, \ x \in X \quad \text{for all } p \in [0,1]$$

to represent elements of $\mathcal{D}X$

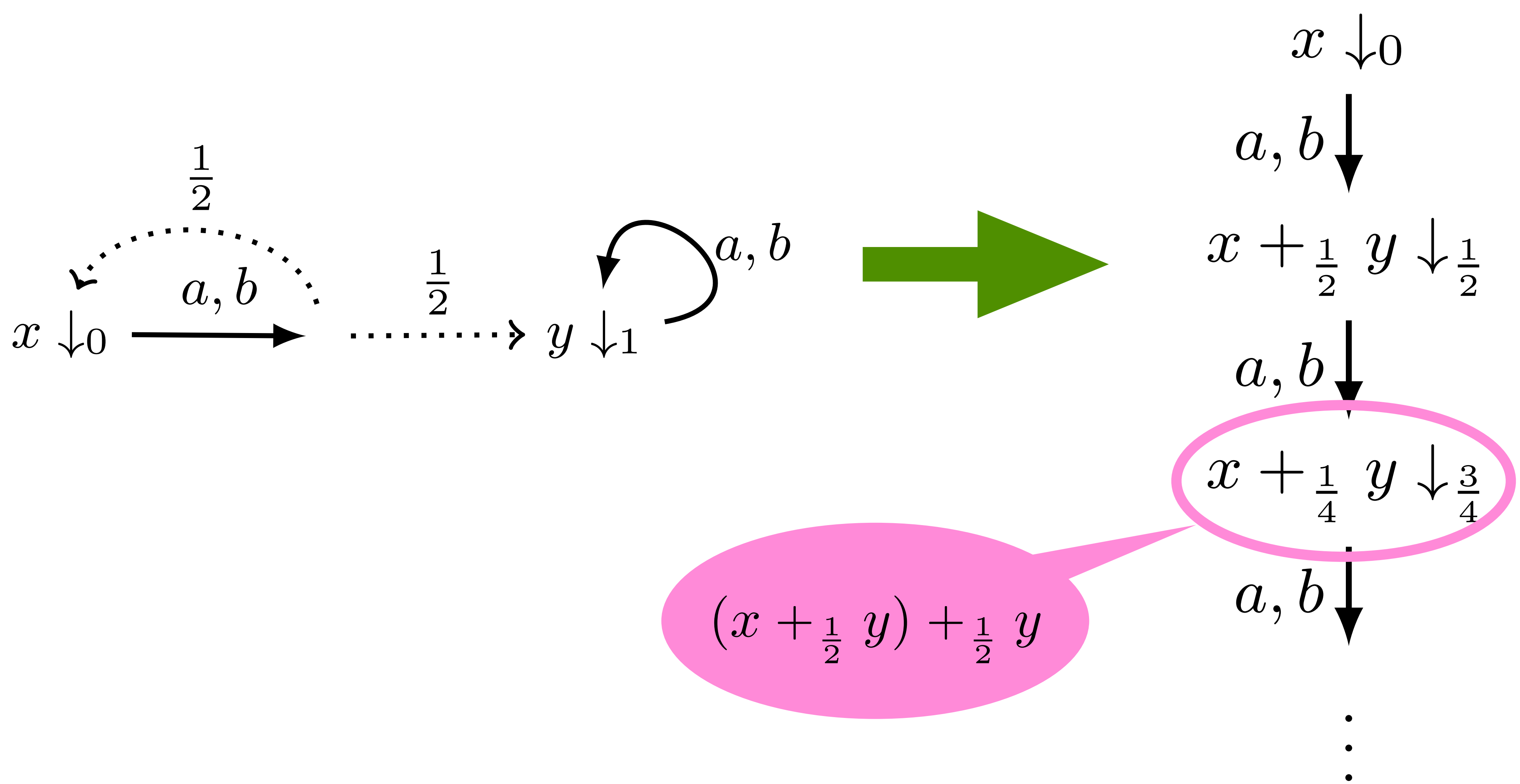
GSOS Semantics for Nondeterministic Automata

$$\frac{-}{\star \xrightarrow{a} \star} \qquad \frac{s \xrightarrow{a} s' \quad t \xrightarrow{a} t'}{s \oplus t \xrightarrow{a} s' \oplus t'} \qquad \frac{-}{\star \downarrow_0} \qquad \frac{s \downarrow_{b_1} \quad t \downarrow_{b_2}}{s \oplus t \downarrow_{b_1 \sqcup b_2}}$$



GSOS Semantics for Probabilistic Automata

$$\frac{s \xrightarrow{a} s' \qquad t \xrightarrow{a} t'}{s +_p t \xrightarrow{a} s' +_p t'} \qquad \frac{s \downarrow_{q_1} \qquad t \downarrow_{q_2}}{s +_p t \downarrow_{p \cdot q_1 + (1-p) \cdot q_2}}$$



The Algebraic Theory of Semilattices with Bottom

$$s, t \ ::= \ \star, \ s \oplus t, \ x \in X$$

$$\begin{array}{ccc}
(x \oplus y) \oplus z & \overset{(A)}{=} & x \oplus (y \oplus z) \\
x \oplus y & \overset{(C)}{=} & y \oplus x \\
x \oplus x & \overset{(I)}{=} & x \\
x \oplus \star & \overset{(B)}{=} & x
\end{array}$$

The set of terms quotiented by these axioms is isomorphic to $\mathcal{P}X$

this theory is a presentation for the powerset monad

The Algebraic Theory of Convex Algebras

$$s, t \ ::= \ s +_p t, \ x \in X \quad \text{for all } p \in [0,1]$$

$$\begin{array}{ccc}
(x +_q y) +_p z & \stackrel{(A_p)}{=} & x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z) \\
x +_p y & \stackrel{(C_p)}{=} & y +_{1-p} x \\
x +_p x & \stackrel{(I_p)}{=} & x
\end{array}$$

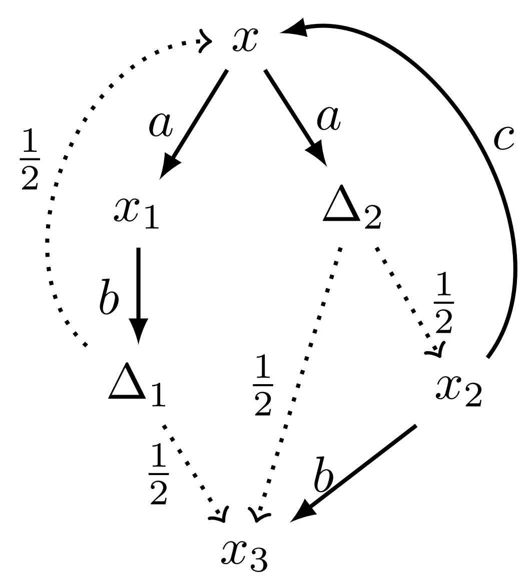
The set of terms quotiented by these axioms is isomorphic to $\mathcal{D}X$

this theory is a presentation for the distribution monad

Probabilistic Nondeterministic Language Semantics ?

NPA

X → ? x ($\mathcal{P}\mathcal{D}$X)^A



$$[\![x]\!] = ???$$

$$[\![\cdot]\!] : X \to ?^{A^*}$$

Algebraic Theory for Subsets of Distributions ?

For our approach it is convenient to have a theory presenting subsets of distributions

Monads can be composed by means of distributive laws, but, unfortunately, there exists no distributive law amongst powerset and distributions (Daniele Varacca Ph.D thesis)

Other general approach to compose monads/algebraic theories fail

Our first step is to decompose the powerset monad...

Three Algebraic Theories

Nondeterminism $\bigoplus$

$$(x \oplus y) \oplus z \overset{(A)}{=} x \oplus (y \oplus z)$$
$$x \oplus y \overset{(C)}{=} y \oplus x$$
$$x \oplus x \overset{(I)}{=} x$$

Monad: $\mathcal{P}_{ne}$

Algebras: **Semilattices**

Probability $+_p$

$$(x +_q y) +_p z \overset{(A_p)}{=} x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z)$$
$$x +_p y \overset{(C_p)}{=} y +_{1-p} x$$
$$x +_p x \overset{(I_p)}{=} x$$

Monad: $\mathcal{D}$

Algebras: **Convex Algebras**

Termination $\star$

no axioms

Monad: $\cdot + 1$

Algebras: **Pointed Sets**

The Algebraic Theory of Convex Semilattices

$$\oplus \quad +_p$$

$$(x \oplus y) \oplus z \overset{(A)}{=} x \oplus (y \oplus z) \qquad (x +_q y) +_p z \overset{(A_p)}{=} x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z)$$

$$x \oplus y \overset{(C)}{=} y \oplus x \qquad x +_p y \overset{(C_p)}{=} y +_{1-p} x$$

$$x \oplus x \overset{(I)}{=} x \qquad x +_p x \overset{(I_p)}{=} x$$

$$(x \oplus y) +_p z \overset{(D)}{=} (x +_p z) \oplus (y +_p z)$$

Monad C: non-empty convex subsets of distributions

One proof is more semantic: the strategy is rather standard but the full proof is tough

convexity comes from the following derived law

$$s \oplus t \overset{(C)}{=} s \oplus t \oplus s +_p t$$

One proof is more syntactic: based on normal form and a unique base theorem. Hope to be generalised by more abstract categorical machinery

Adding Termination

$$\oplus \quad +_p \quad \star$$

$$(x \oplus y) \oplus z \stackrel{(A)}{=} x \oplus (y \oplus z) \qquad (x +_q y) +_p z \stackrel{(A_p)}{=} x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z)$$

$$x \oplus y \stackrel{(C)}{=} y \oplus x \qquad x +_p y \stackrel{(C_p)}{=} y +_{1-p} x$$

$$x \oplus x \stackrel{(I)}{=} x \qquad x +_p x \stackrel{(I_p)}{=} x$$

$$(x \oplus y) +_p z \stackrel{(D)}{=} (x +_p z) \oplus (y +_p z)$$

The Algebraic Theory of Pointed Convex Semilattices

$$x \oplus \star \stackrel{(B)}{=} x$$

The Algebraic Theory of Convex Semilattices with Bottom

$$x \oplus \star \stackrel{(T)}{=} \star$$

The Algebraic Theory of Convex Semilattices with Top

These three algebras are those freely generated by the singleton set 1

They give rise to three different semantics: may, must, and may-must

$$\mathbb{M}_{\mathcal{I}} = (\mathcal{I}, \text{min-max}, +_p^{\mathcal{I}}, [0,0])$$

$$\mathcal{I} = \{[x,y] \mid x, y \in [0,1] \text{ and } x \leq y\}$$

$$\text{min-max}([x_1,y_1],[x_2,y_2]) = [\min(x_1,x_2), \max(y_1,y_2)]$$

$$[x_1,y_1] +_p^{\mathcal{I}} [x_2,y_2] = [x_1 +_p x_2,\ y_1 +_p y_2]$$

The Theory of Pointed Convex Semilattices

$$\mathbb{M}\text{ax} = \quad ([0,1], \max, +_p, 0)$$

The Algebraic Theory of Convex Semilattices with bottom

$$\mathbb{M}\text{in} = ([0,1], \min, +_p, 0)$$

The Algebraic Theory of Convex Semilattices with Top

Syntax and Transitions

For the three semantics, we use the same syntax

$$s, t \ ::= \ \star, \ s \oplus t, \ s +_p t, \ x \in X \quad \text{for all } p \in [0,1]$$

and transitions

$$\frac{-}{\star \xrightarrow{a} \star} \qquad \frac{s \xrightarrow{a} s' \quad t \xrightarrow{a} t'}{s \oplus t \xrightarrow{a} s' \oplus t'} \qquad \frac{s \xrightarrow{a} s' \quad t \xrightarrow{a} t'}{s +_p t \xrightarrow{a} s' +_p t'}$$

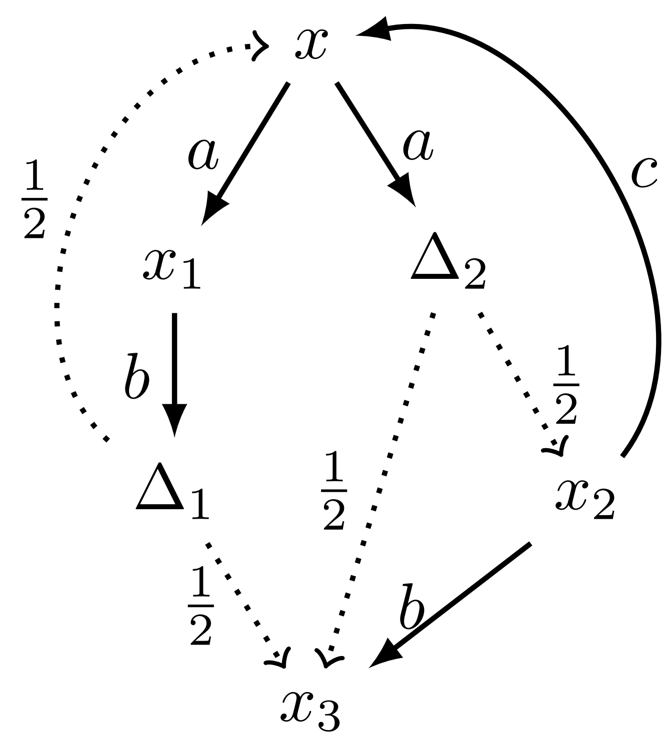
but different output functions...

Example without outputs

$$x \xrightarrow{a} x_1 \oplus (x_3 +_{\frac{1}{2}} x_2)$$

$$x_1 \xrightarrow{b} x +_{\frac{1}{2}} x_3$$

$$x_2 \xrightarrow{b} x_3 \qquad x_2 \xrightarrow{c} x$$



$$x \xrightarrow{b,c} \star$$

$$x_1 \xrightarrow{a,c} \star$$

$$x_2 \xrightarrow{a} \star$$

$$x_3 \xrightarrow{a,b,c} \star$$

$$x \xrightarrow{a} x_1 \oplus (x_3 +_{\frac{1}{2}} x_2) \xrightarrow{b} (x +_{\frac{1}{2}} x_3) \oplus (\star +_{\frac{1}{2}} x_3)$$

Outputs for May

We take as algebra of outputs

$$\mathbb{M}\mathrm{ax} = ([0,1], \max, +_p, 0)$$

that gives rise to the following three rules

$$\frac{-}{\star \downarrow_0} \qquad \frac{s \downarrow_{q_1} \quad t \downarrow_{q_2}}{s \oplus t \downarrow_{\mathtt{max}(q_1, q_2)}} \qquad \frac{s \downarrow_{q_1} \quad t \downarrow_{q_2}}{s +_p t \downarrow_{q_1 +_p q_2}}$$

Outputs for Must

We take as algebra of outputs

$$\mathbb{M}\text{in} = ([0,1], \min, +_p, 0)$$

that gives rise to the following three rules

$$\frac{-}{\star \downarrow_0} \qquad \frac{s \downarrow_{q_1} \quad t \downarrow_{q_2}}{s \oplus t \downarrow_{\mathtt{min}(q_1, q_2)}} \qquad \frac{s \downarrow_{q_1} \quad t \downarrow_{q_2}}{s +_p t \downarrow_{q_1 +_p q_2}}$$

Outputs for May-Must

We take as algebra of outputs

$$\mathbb{M}_{\mathcal{I}} = (\mathcal{I}, \text{min-max}, +_p^{\mathcal{I}}, [0,0])$$

that gives rise to the following three rules

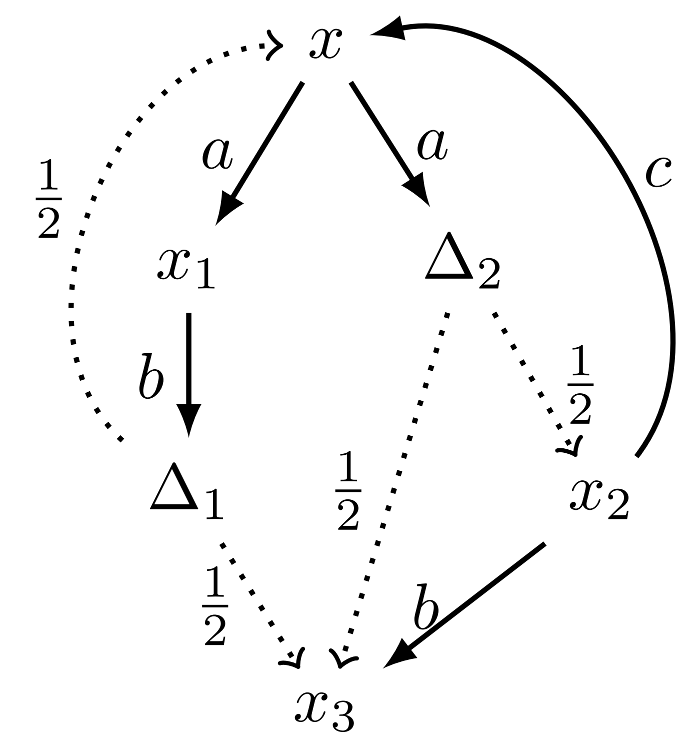
$$\frac{-}{\star \downarrow_{[0,0]}} \qquad \frac{s \downarrow_I \quad t \downarrow_J}{s \oplus t \downarrow_{\texttt{min-max}(I,J)}} \qquad \frac{s \downarrow_I \quad t \downarrow_J}{s +_p t \downarrow_{I +_p^{\mathcal{I}} J}}$$

Example with outputs

$x \xrightarrow{a} x_1 \oplus (x_3 +_{\frac{1}{2}} x_2)$

$x_1 \xrightarrow{b} x +_{\frac{1}{2}} x_3$

$x_2 \xrightarrow{b} x_3 \qquad x_2 \xrightarrow{c} x$



$x \xrightarrow{b,c} \star$

$x_1 \xrightarrow{a,c} \star$

$x_2 \xrightarrow{a} \star$

$x_3 \xrightarrow{a,b,c} \star$

All states output 1

$x \downarrow_1 \qquad x_1 \downarrow_1 \qquad x_2 \downarrow_1 \qquad x_3 \downarrow_1$

May $\quad x \downarrow_1 \xrightarrow{a} x_1 \oplus (x_3 +_{\frac{1}{2}} x_2) \downarrow_1 \xrightarrow{b} (x +_{\frac{1}{2}} x_3) \oplus (\star +_{\frac{1}{2}} x_3) \downarrow_1$

Must $\quad x \downarrow_1 \xrightarrow{a} x_1 \oplus (x_3 +_{\frac{1}{2}} x_2) \downarrow_1 \xrightarrow{b} (x +_{\frac{1}{2}} x_3) \oplus (\star +_{\frac{1}{2}} x_3) \downarrow_{\frac{1}{2}}$

Conclusions

Traces carry a convex semilattice
The three trace semantics are convex semilattice homomorphisms
Trace equivalences are congruence w.r.t. convex semilattice operations
Coinduction up-to these operation is sound

Both probabilistic and convex bisimilarity
implies the three trace equivalences

The equivalences are "backward compatible" with standard trace equivalences
for non deterministic and probabilistic systems

The may-equivalence coincides with one in
Bernardo, De Nicola, Loreti TCS 2014


Thank You